# Problem 2.5

Find the electric field a distance $z$ above the center of a circular loop of radius $r$ (Fig. 2.9) that carries a uniform line charge $\lambda$.

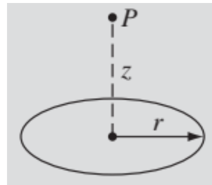


Fig. 2.9

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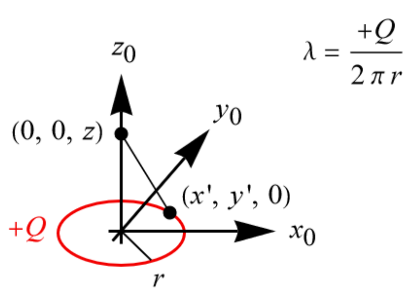
### Solution

Start by drawing a schematic for some point on the circular loop.




The formula for the electric field from a continuous distribution of charge along a line is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dl' = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)dl'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dl',$$

where the integral is taken over the line where the charge exists. Note that $\mathbf{r}$ is the position vector to where we want to know the electric field, $\mathbf{r}'$ is the position vector to the point we chose on the line, and $\mathscr{r} = |\mathbf{r}-\mathbf{r}'|$ is the distance from the point we chose on the line to where we want to know the electric field.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{x_0^2+y_0^2=r^2} \frac{\lambda}{\left[\sqrt{(0-x')^2+(0-y')^2+(z-0)^2}\right]^3}(\langle 0,0,z\rangle - \langle x',y',0\rangle)\,ds'$$

The loop is circular, so the appropriate parameterization is done with polar coordinates.

$$\mathbf{r}' = r\langle\cos\theta',\sin\theta',0\rangle, \quad 0 \le \theta' \le 2\pi$$

Consequently, the electric field at $\mathbf{r} = \langle 0,0,z\rangle$ is

$$\mathbf{E} = \frac{\lambda}{4\pi\epsilon_0}\int_0^{2\pi} \frac{1}{\left[\sqrt{(0-r\cos\theta')^2+(0-r\sin\theta')^2+(z-0)^2}\right]^3}(\langle 0,0,z\rangle - r\langle\cos\theta',\sin\theta',0\rangle)(r\,d\theta').$$

Simplify the integrand and then integrate the components.

$$\begin{aligned}
\mathbf{E} &= \frac{\lambda}{4\pi\epsilon_0}\int_0^{2\pi}\frac{1}{(r^2+z^2)^{3/2}}\langle -r\cos\theta', -r\sin\theta', z\rangle(r\,d\theta') \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{r}{(r^2+z^2)^{3/2}}\left\langle -r\int_0^{2\pi}\cos\theta'\,d\theta', -r\int_0^{2\pi}\sin\theta'\,d\theta', z\int_0^{2\pi}d\theta'\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{r}{(r^2+z^2)^{3/2}}\langle -r(0), -r(0), z(2\pi)\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{r}{(r^2+z^2)^{3/2}}\langle 0, 0, 2\pi z\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{2\pi r z}{(r^2+z^2)^{3/2}}\langle 0, 0, 1\rangle
\end{aligned}$$

Therefore, the electric field at $\mathbf{r} = \langle 0, 0, z\rangle$ is

$$\boxed{\mathbf{E} = \frac{\lambda}{2\epsilon_0}\frac{rz}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}.}$$

Observe that

$$\begin{aligned}
\lim_{r\to 0}\mathbf{E} &= \lim_{r\to 0}\frac{\lambda}{2\epsilon_0}\frac{rz}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}} = \frac{\lambda}{2\epsilon_0}\frac{(0)z}{(0^2+z^2)^{3/2}}\hat{\mathbf{z}} = \mathbf{0} \\
\lim_{z\to 0}\mathbf{E} &= \lim_{z\to 0}\frac{\lambda}{2\epsilon_0}\frac{rz}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}} = \frac{\lambda}{2\epsilon_0}\frac{r(0)}{(r^2+0^2)^{3/2}}\hat{\mathbf{z}} = \mathbf{0}.
\end{aligned}$$

In order to see what happens if $z \gg r$, rewrite the formula so that each term is a ratio of $r$ and $z$, $z$ being in the denominator, and use the binomial theorem.

$$\begin{aligned}
\mathbf{E} &= \frac{\lambda}{2\epsilon_0}\frac{rz}{\left[z^2\left(\frac{r^2}{z^2}+1\right)\right]^{3/2}}\hat{\mathbf{z}} \\
&= \frac{\lambda}{2\epsilon_0}\frac{rz}{z^3\left(\frac{r^2}{z^2}+1\right)^{3/2}}\hat{\mathbf{z}} \\
&= \frac{\lambda}{2\epsilon_0}\frac{r}{z^2}\left(1+\frac{r^2}{z^2}\right)^{-3/2}\hat{\mathbf{z}} \\
&= \frac{\lambda}{2\epsilon_0}\frac{r}{z^2}\left[\sum_{k=0}^{\infty}\frac{\Gamma\left(-\frac{3}{2}+1\right)}{\Gamma(k+1)\Gamma\left(-\frac{3}{2}-k+1\right)}\left(\frac{r^2}{z^2}\right)^k\right]\hat{\mathbf{z}} \\
&= \frac{\lambda}{2\epsilon_0}\frac{r}{z^2}\left[\sum_{k=0}^{\infty}\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(k+1)\Gamma\left(-\frac{1}{2}-k\right)}\left(\frac{r}{z}\right)^{2k}\right]\hat{\mathbf{z}} \\
&= \frac{\lambda}{2\epsilon_0}\frac{r}{z^2}\left[\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(1)\Gamma\left(-\frac{1}{2}\right)}\left(\frac{r}{z}\right)^0 + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(2)\Gamma\left(-\frac{3}{2}\right)}\left(\frac{r}{z}\right)^2 + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(3)\Gamma\left(-\frac{5}{2}\right)}\left(\frac{r}{z}\right)^4 + \cdots\right]\hat{\mathbf{z}}
\end{aligned}$$

Continue the simplification.

$$\mathbf{E} = \frac{\lambda}{2\epsilon_0}\frac{r}{z^2}\left[1 - \frac{3}{2}\left(\frac{r}{z}\right)^2 + \frac{15}{8}\left(\frac{r}{z}\right)^4 - \cdots\right]\hat{\mathbf{z}}$$

$$= \frac{Q}{2(2\pi)\epsilon_0}\frac{1}{z^2}\left(1 - \frac{3r^2}{2z^2} + \frac{15r^4}{8z^4} - \cdots\right)\hat{\mathbf{z}}$$

If $z \gg r$, then $r^2/z^2$ and all higher-order terms are so much smaller than 1 that they can be neglected.

$$\mathbf{E} \approx \frac{Q}{2(2\pi)\epsilon_0}\frac{1}{z^2}\hat{\mathbf{z}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}\hat{\mathbf{z}}$$

The lesson is that far away from the circular loop the electric field is the same as if it were a point charge.